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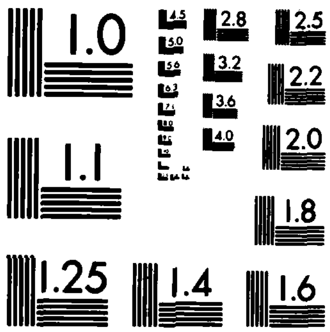

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ON A NONLINEAR EIGENVALUE PROBLEM OCCURRING IN POPULATION GENETICS

Ph. Clément and L. A. Peletier

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

We discuss the nonlinear eigenvalue problem

$$(P)\qquad \begin{cases} u'' + \lambda f(x,u) = 0 & -1 < x < 1,\ \lambda \geq 0 \\ u'(-1) = u'(+1) = 0, & 0 \leq u(x) \leq 1 \end{cases}$$

where

$$f(x,u) = u(1-u)[u - a(x)]$$

and

$$a(x) = \frac{1}{2}[1 - \varepsilon r(x) + h] \qquad \varepsilon \geq 0,\quad h \in \mathbb{R}$$

with $r(-x) = -r(x)$ and $r' \geq 0$.

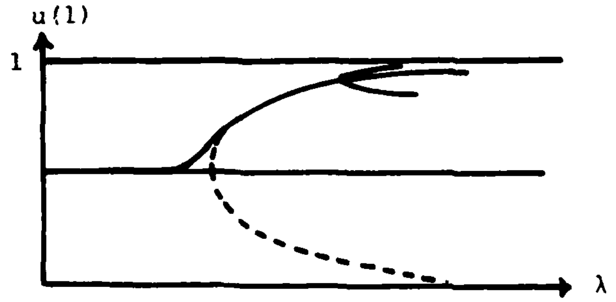
For $\varepsilon = h = 0$ the solution of Problem P is well known, and every solution, except $u = 0$ and $u = 1$ is unstable with respect to the corresponding parabolic problem.

We show how the branch of increasing solutions changes as ε becomes positive, and acquires a bifurcation point $(\bar{\lambda},\bar{u})$ beyond which this branch becomes stable. If h becomes nonzero as well, this bifurcation point is shown to break up.

As an illustration we consider an example in which the branch of increasing solutions can be computed. Here

$$f(x,u) = \begin{cases} -u & 0 \leq x < a(x) \\ 1 - u & a(x) < u \leq 1 \end{cases}$$

where $a(x)$ is given above.



$\varepsilon > 0,\ h = 0$

AMS (MOS) Subject Classifications: 34B15, 35K55
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*Delft University of Technology, Delft, The Netherlands.
**University of Leiden, Leiden, The Netherlands.

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SIGNIFICANCE AND EXPLANATION

Consider a population, consisting of three genotypes: AA, Aa and aa, which tend to disperse through a habitat Ω and possess different survival fitnesses. Let $u(x,t)$ denote the fraction of allele A amongst the total number of alleles in the population at a point x in the habitat, at time t. Then, according to a model due to Fisher and Haldane, the evolution of u is described by the nonlinear diffusion equation

$$u_t = D\Delta u + f(x,u) \qquad x \in \Omega, \quad t > 0$$

in which D is a positive constant, measuring the rate of dispersal and f a function determined by the relative fitness of the genotypes. If the fitness of the heterozygote Aa is lower than that of either of the homozygotes, then f is of the form given in the Abstract. When $\Omega = (-1,1)$ and we assume no-flux boundary conditions we arrive at Problem P, with $\lambda = 1/D$.

In this paper we shall assume $f_x \geq 0$, and study the set of nondecreasing equilibrium solutions, and their stability properties. In particular, it is shown that if an equilibrium solution is decreasing on some interval, then it must necessarily be unstable.

Problem P contains three parameters, λ: measuring the relative importance of dispersal and selection, ε: measuring the gradients in the selection pressure f_x and h: measuring the departure from the symmetry property

$$(*) \qquad f(-x,1-u) = -f(x,u) \qquad x \in (-1,1), \quad u \in [0,1] .$$

We find that if $(*)$ is satisfied $(h = 0)$, and $f_x \not\equiv 0$, there exists a branch of increasing solutions $\phi(x,\lambda)$ such that

$$\phi(-x,\lambda) = 1 - \phi(x,\lambda) \qquad x \in [-1,1] ,$$

which is unstable for λ small and stable for λ large, the exchange of stability being achieved at a bifurcation point on this branch. This singularity disappears when h becomes nonzero, and the symmetry relation is no longer valid. Biologically, this means that for small dispersal rates, stable equilibrium solutions exist, which mirror the selective advantages in the habitat.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON A NONLINEAR EIGENVALUE PROBLEM
OCCURRING IN POPULATION GENETICS

Ph. Clément * and L.A. Peletier **

1. INTRODUCTION

In a previous paper [8] the second author considered the following nonlinear eigenvalue problem

$$(P) \qquad \begin{cases} u'' + \lambda f(x,u) = 0 & -1 < x < 1 \ , \ \lambda \geq 0 \ . \\ u'(-1) = u'(1) = 0 \end{cases}$$

where

$$(1.1) \qquad f(x,u) = u(1-u)(u-a(x))$$

and a satisfies

A1 $a \in C^1\left([-1,1]\right)$

A2 $0 < a(x) < 1$ for $-1 \leq x \leq 1$.

This problem arises in population genetics as a simple diffusion model for the propagation of genetic material in a population, see [5,6]. The solutions of (P) are stationary solutions of the problem

*Delft University of Technology, Delft, The Netherlands.

**University of Leiden, Leiden, The Netherlands.

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$$\text{(D)} \qquad \begin{cases} u_t = u_{xx} + \lambda f(x,u) & -1 < x < 1 \ , \ t > 0 \\ u_x(t,-1) = u_x(t,1) = 0 & t \geq 0 \ . \end{cases}$$

Since u represents a fraction of alleles in the population, we are only interested in solutions u taking values in $[0,1]$; $u \equiv 0$ and $u \equiv 1$ are (trivial) solutions of (P), and they are both asymptotically stable. As a consequence of the implicit function theorem one easily proves that the branches $\{(\lambda,0)|\lambda \in \mathbb{R}^+\}$ and $\{(\lambda,1)|\lambda \in \mathbb{R}^+\}$ do not have a bifurcation point in $\mathbb{R}^+ \times C([-1,1])$ where $C([-1,1])$ denotes the space of continuous functions on $[-1,1]$ equipped with the supremum norm.

When $\underline{a'(x) = 0}$, $x \in [-1,1]$, the trivial solutions are the only stable ones [1].

When $\underline{a'(x) < 0}$, $x \in [-1,1]$, it was proved in [7,Theorem 4] and [5] that for sufficiently large values of λ , there does exist at least one non-trivial solution which is stable. (We prove in Appendix B that stable solutions have to be increasing).

In this paper, we are interested in the question as to how these stable solutions emerge when $a(x)$ changes from a constant to a non-constant function. To be specific we shall write

$$\text{(1.2)} \qquad a(x) = \tfrac{1}{2}\Big(1-\varepsilon r(x)+h\Big) \ , \qquad \varepsilon,h \in \mathbb{R}$$

where $r(-x) = -r(x)$, $x \in [-1,1]$.

In the example given in [8], where $h = 0$, $r(x) = 1$ on $(0,1]^*$ and $\varepsilon \in [0,1)$, it was shown that the stable solutions are part of continuum in $\mathbb{R}^+ \times C([-1,1])$ which extends all the way back to $\lambda = 0$, and that (λ,u) converges to $(0,\frac{1}{2})$ as $\lambda \downarrow 0$. In addition it was shown in [8], that, under the assumptions A1, A2, for λ small, there exists a unique curve C of nontrivial unstable solutions

* For this choice of r the function a does not satisfy the condition A1, but this has no effect on the result.

$u(\lambda)$, bifurcating from $(0,\alpha)$ where

$$(1.3) \qquad \alpha = \tfrac{1}{2} \int_{-1}^{+1} a(x)dx \ .$$

Thus, in the example considered above, in which f has the symmetry property:

$$(1.4) \qquad f(-x,1-u) = -f(x,u) \qquad x \in [-1,1] \ , \ u \in [0,1] \ ,$$

(that is f is antisymmetric with respect to the point $(0,\tfrac{1}{2})$ in the (x,u)-plane), the branch of nontrivial solutions C extends to infinity and eventually the solutions of this branch become stable. As a consequence it was shown that C contains a bifurcation point $(\bar{\lambda},\bar{u})$.

In this paper we shall extend these results to more general functions r , and discuss the dependence of $(\bar{\lambda},\bar{u})$ on ε .

If h becomes non zero and f no longer has the symmetry property (1.4), we prove some partial results which suggest that the continuum C breaks up at the point $(\bar{\lambda},\bar{u})$. We conjecture on the basis of these results that when f fails to have the symmetry property (1.4), then C does not contain any stable solution.

We conclude this paper with an example of a different type of function f :

$$f(x,u) = \begin{cases} -u & 0 \le u < a(x) \\ 1-u & a(x) < u \le 1 \ , \end{cases}$$

where $a(x)$ is given by (1.2) and $r(x) = x$. In this example the branch C can be computed explicitely, both when h is zero and when h is not zero.

We gratefully acknowledge S. Hastings for suggesting this example.

2. THE CONTINUUM C

Define the function $J(u) = \int_{-1}^{+1} f(x,u)dx$, where f is given by (1.1). Then

(2.1) $$J(u) = 0 \Longleftrightarrow u \in \{0,\alpha,1\}$$

where α is defined in (1.3). Recall that by A2, $0 < \alpha < 1$. In this section we shall show that from the point $(0,\alpha)$ emanates a branch of nontrivial solutions of (P). We shall denote it by C. By a solution of Problem (P) we shall mean a pair $(\lambda,u) \in \mathbb{R}^+ \times C^2([-1,1])$, which satisfies (P), where $C^2([-1,1])$ is the space of twice continuously differentiable functions on $[-1,1]$ equipped with the usual norm.

THEOREM 1. *Suppose* a *satisfies* A1 *and* A2. *Then there exists a maximal connected set of nontrivial solutions* C *in* $\mathbb{R}^+ \in C^2([-1,1])$, *whose projection on* $\mathbb{R}^+$ *is unbounded, with the properties:*

(i) $\lim\limits_{\substack{(\lambda,u)\in C \\ \lambda\downarrow 0}}(\lambda,u) = (0,\alpha)$ *in* $\mathbb{R} \times C^2([-1,1])$.

(ii) *There exists* $\lambda_0 > 0$ *such that for* $\lambda \in (0,\lambda_0)$ *there exists one and only one nontrivial solution* (λ,u) , *and* (λ,u) *belongs to* C .

(iii) *If* $(\lambda,u) \in C$, *then* $0 < u(x) < 1$, $x \in [-1,1]$

(iv) *If* a *satisfies in addition*

A3 $a'(x) \le 0 \quad x \in [-1,1]$,

a *not a constant, then* $(\lambda,u) \in C$ *implies* $u'(x) > 0$, $x \in (-1,1)$. (u *is a "cline").*

REMARK. For $\lambda \in (0,\lambda_0)$, the function $\lambda \to u(\lambda) \in C^2([-1,1])$ is analytic.

PROOF. To prove (i) and (ii) we shall rewrite Problem (P) such that we can apply Theorem A from the Appendix. Observe that if (λ,u) is a solution of Problem (P) and $\lambda > 0$, then $\int_{-1}^{+1} f(x,u(x))dx = 0$. Therefore solutions of Problem (P) with $\lambda > 0$ are also solutions of Problem (P1) and conversely:

$$\text{(P1)} \qquad \begin{cases} -u'' = \lambda\{f(x,u) - \frac{1}{2}\int_{-1}^{+1} f(x,u(x))dx\} \\ u'(\pm 1) = 0 \\ \int_{-1}^{+1} f(x,u(x))dx = 0 \,. \end{cases}$$

Let k be such that if $g \in C([-1,1])$ with $\int_{-1}^{+1} g(x)dx = 0$ and $u(x) := \int_{-1}^{+1} k(x,y)g(y)dy$, then u satisfies (i) $\int_{-1}^{+1} u(x)dx = 0$ and (ii) $-u'' = g$, $x \in (-1,+1)$ and $u'(\pm 1) = 0$. Let β denote $\frac{1}{2}\int_{-1}^{+1} u(x)dx$ and $\gamma = \beta - \alpha$, where α is defined in (1.3), and $v = u - \beta$, Then we can rewrite Problem (P1) as

$$\text{(P2)} \qquad \begin{cases} v(x) = \lambda \int_{-1}^{+1} k(x,y)f(y,v(y)+\alpha+\gamma)dy \\ \int_{-1}^{+1} f(x,v(x)+\alpha+\gamma)dx = 0 \,. \end{cases}$$

Now we define the Banach space

$$E := \left\{v \in C([-1,1]) \,\middle|\, \int_{-1}^{+1} v(x)dx = 0\right\} \times \mathbb{R}$$

equipped with the norm $\|(v,\gamma)\| = \max_{x\in[-1,1]} |v(x)| + |\gamma|$, and the map

$$F := \mathbb{R} \times E \to E \quad \text{by}$$

$$F(\lambda;(v,\gamma)) = \begin{cases} v - \lambda \int_{-1}^{+1} k(\cdot,y)f(y,v(y)+\alpha+\gamma)dy \\ \int_{-1}^{+1} f(x,v(x)+\alpha+\gamma)dx \,. \end{cases}$$

Plainly Problem (P2) can be rewritten as

$$\text{(P3)} \qquad F(\lambda;(v,\gamma)) = 0 \ , \ \lambda \in \mathbb{R} \ , \ (v,\gamma) \in E \ .$$

We are now in a position to apply Theorem A of the Appendix and we shall prove that the conditions a) to f) are all satisfied. a) follows from the definition of F and α. b), c) and f) with F analytic are standard. To prove d), one computes the partial Fréchet derivative of F with respect to (v,γ) at $(0,(0,0))$.

$$F'_{(v,\gamma)}(0;(0,0))(\hat{v},\hat{\gamma}) = \begin{cases} \hat{v} \\ \int_{-1}^{+1} f_u(x,\alpha)[\hat{v}(x)+\hat{\gamma}]dx \ , \end{cases}$$

where $(\hat{v},\hat{\gamma}) \in E$. Since $\int_{-1}^{+1} f_u(x,\alpha)dx = 2\alpha(1-\alpha) \neq 0$ it follows that $F'_{(v,\gamma)}(0,(0,0)) \in \mathrm{Isom}(E,E)$. It remains to prove e). First observe that as a consequence of (2.1), $F(0,(v,\gamma)) = 0$ if and only if $v = 0$ and $\gamma \in \{-\alpha,0,1-\alpha\}$. Then one observes that the component of solutions in $\mathbb{R} \times E$ which contains $(0,-\alpha)$ (resp. $(0,1-\alpha)$) is exactly $\{(\lambda,-\alpha))\lambda\in\mathbb{R}\}$ (resp. $\{(\lambda,1-\alpha)|\lambda\in\mathbb{R}\}$ since there is no bifurcation on these lines. Thus the component $\mathcal{D}$ of solutions in $\mathbb{R} \times E$ which contains $(0,0)$ does not meet $\{0\} \times E$, and condition e) is satisfied. It follows from Theorem A that Problem (P3) possess a maximal connected set of solutions in $\mathbb{R}^+ \times E$ which is unbounded in $\mathbb{R}^+ \times E$ and such that

$$\lim_{\substack{(\lambda,(v,\gamma))\in\mathcal{D} \\ \lambda\downarrow 0}} (\lambda,(v,\gamma)) = (0,(0,0)) \ .$$

It is easy to verify that $v \in C^2([-1,1])$. Next we define $C := \{(\lambda,u)\in\mathbb{R}^+\times C^2([-1,1])|u=v+\gamma+\alpha \text{ with } (\lambda,(v,\gamma))\in\mathcal{D}\}$. C is an unbounded maximal connected set in $\mathbb{R}^+ \times C([-1,1])$; moreover $(\lambda,u) \in C$ are nontrivial solutions of (P). Thus since the $\mathbb{R}\times C([-1,1])$ and the $\mathbb{R}\times C^2([-1,1])$ topology are the same on the set of solutions of (P), C is a maximal connected set in $\mathbb{R}^+\times C^2([-1,1])$, such that $\lim_{\substack{(\lambda,u)\in C\\ \lambda\downarrow 0}}(\lambda,u) = (0,\alpha)$ in $\mathbb{R}\times C^2([-1,1])$.

This proves part (i) of Theorem 1.

Part (ii) follows from f) of Theorem A. Next we prove that $(\lambda,u)\in C$ implies $0 < u(x) < 1$, $x\in[-1,1]$. It is sufficient to prove $\max_{x\in[-1,1]}|u(x)-\frac{1}{2}| < \frac{1}{2}$ for $(\lambda,u)\in C$. First observe that for λ small, $\max_{x\in[-1,1]}|u(x)-\frac{1}{2}| < \frac{1}{2}$, with $(\lambda,u)\in C$. Assume that for some $(\lambda,u)\in C$, $\max_{x\in[-1,1]}|u(x)-\frac{1}{2}| = \frac{1}{2}$. From the boundary conditions, it follows that there exists $\xi\in[-1,1]$ such that $u'(\xi) = 0$, $u(\xi) = 0$ or 1. But the function $\bar{u}(x)\equiv 0$ or $\equiv 1$ would be a solution of the initial value problem

$$(\mathrm{IP})\begin{cases} -v'' = \lambda f(x,v) \\ v(\xi) = u(\xi) \\ v'(\xi) = 0 \end{cases}$$

as well as the function u which is not constant. This contradicts the uniqueness of the initial value problem (IP). We have then that $(\lambda,u)\in C$ implies $\max_{x\in[-1,1]}|u(x)-\frac{1}{2}| = \frac{1}{2}$. One concludes by observing that the function $(\lambda,u)\in C \to \max_{x\in[-1,1]}|u(x)-\frac{1}{2}|$ is continuous from $\mathbb{R}\times C([-1,1])$ into $\mathbb{R}$ and that C is connected in $\mathbb{R}\times C([-1,1])$. This proves part (iii).

As a consequence we prove next that $\mathrm{Proj}_{\mathbb{R}^+} C$ is unbounded. We have $0 < v(x)+\gamma+\alpha < 1$ for all $(\lambda,(v,\gamma))\in \mathcal{D}$. Since $\int_{-1}^{+1} v(x)dx = 0$, we also have $0 < \gamma+\alpha < 1$, for all $(\lambda,(v,\gamma))\in\mathcal{D}$ and thus $\sup_{(\lambda,(v,\gamma))\in\mathcal{D}}\|(v,\gamma)\| < \infty$, hence $\mathrm{Proj}_{\mathbb{R}^+}\mathcal{D} = \mathrm{Proj}_{\mathbb{R}^+}$ is bounded, since $\mathcal{D}$ is unbounded in $\mathbb{R}^+\times E$.

Finally we prove (iv). For $(\lambda,u) \in C$, we define $\mu_D(\lambda,u)$ to be the principal eigenvalue of the problem

$$\begin{cases} -h'' - \lambda f_u(.,u)h = \mu h & -1 < x < 1 \\ h(-1) = h(+1) = 0 . \end{cases}$$

For $(\lambda,u) \in C$ and λ small it is easy to see that $\mu_D(\lambda,u) > 0$. But $\mu_D(\lambda,u) \neq 0$ when $(\lambda,u) \in C$. Indeed $w = u'$ satisfies:

$$(2.2) \qquad \begin{cases} -w'' - \lambda f_u w = \lambda f_x = \lambda u(1-u)(-a') \geq 0 \\ w(-1) = w(1) = 0 \end{cases}$$

Suppose $\mu_D(\lambda,u) = 0$. Then $\int_{-1}^{+1} f_x h dx$ should be zero where h denotes the principal eigenfunction chosen positive and normalized by $\max_{x\in[-1,1]} h(x) = 1$. Since a is not a constant, this is impossible. As in part (iii) we conclude that $\mu_D(\lambda,u) > 0$ on C by observing that μ_D is a continuous function of (λ,u) and C is connected. Since the operator in the left-hand side of (2.2) is coercive, and the right-hand side is nonnegative (not identically zero), then $u'(x) = w(x) > 0$, $x \in (-1,1)$. This completes the proof of Theorem 1.

3. THE SYMMETRIC CASE

Throughout this section we set $h = 0$. Thus

$$a(x) = \tfrac{1}{2}(1-\varepsilon r(x)) ,$$

and we assume that r satisfies

H1 $r(-x) = -r(x) \qquad x \in [-1,1]$

H2 $r'(x) \geq 0$ on $[-1,1]$ and $r'(0) > 0$.

Thus $\alpha = \frac{1}{2}$, and by Theorem 1 the continuum C "emanates" from $(0,\frac{1}{2})$. For convenience we define $v = 2u - 1$. Then Problem (P) is equivalent with

$$(P')\qquad \begin{cases} v'' + \lambda g(x,v) = 0 & -1 < x < 1 \\ v'(-1) = v'(1) = 0 , \end{cases}$$

where

$$g(x,v) = \tfrac{1}{4}(1-v^2)[v+\varepsilon r(x)] .$$

Suppose ϕ^* is a solution of the problem

$$(P'')\qquad \begin{cases} v'' + \lambda g(x,v) = 0 & 0 < x < 1 \\ v(0) = 0 \quad v'(1) = 0 \end{cases}$$

Then due to the sysmmetry property of r given in H1, the function

$$\tilde{\phi}(x) := \begin{cases} -\phi^*(-x) & -1 \le x < 0 \\ \phi^*(x) & 0 \le x \le 1 \end{cases}$$

is a solution of Problem (P'), and $\phi(x) = \frac{1}{2}(1+\tilde{\phi}(x))$ is a solution of Problem (P). Since the function g has the property

$$g(x,v) > g_v(x,v)v \qquad 0 < x \le 1 \ , \ 0 \le v \le 1$$

the following facts about Problem (P") are well-known:

PROPOSITION 2. *Let* r *satisfy hypothesis* H1 *and* H2. *Then*

(i) *for each* $\lambda > 0$, *Problem* (P") *has a unique positive solution* $\phi^*(\lambda)$;

(ii) $\phi^*(\lambda)$ *is increasing and concave;*

(iii) *the map* $\lambda \mapsto \phi^*(\lambda): \mathbb{R}^+ \to C^2([0,1])$ *is analytic*

(iv) $0 \le \lambda_1 < \lambda_2$ *implies* $\phi^*(\lambda_1)(x) < \phi^*(\lambda_2)(x)$ *for* $x \in (0,1]$

(v) $\phi^*(\lambda) \to 0$ *as* $\lambda \to 0$ *in* $C^2([0,1])$.

(vi) $\lim_{\lambda \to \infty} \phi^*(\lambda)(x) = 1$ *for each* $x \in (0,1]$.

They are an easy consequence of the fact that the principal eigenvalue $\mu_1(\lambda)$ of the linearized problem

$$\begin{cases} -h'' - \lambda g_v(x,\phi^*(x))h = \mu h & 0 < x < 1 \\ h(0) = h'(1) = 0 \end{cases}$$

is positive for all $\lambda > 0$.

PROPOSITION 3. *Let* r_1 *and* r_2 *satisfy hypothesis* H1, H2 *and let* ϕ_1^* *and* ϕ_2^* *be the solutions of Problem* (P"), *corresponding to* r_1 *and* r_2 . *Suppose* $r_1 > r_2$ *then* $\phi_1^*(x) > \phi_2^*(x)$ *for* $0 < x \le 1$.

PROOF. Write $f_i(x,u) = u(1-u)(u-a_i(x))$ where $a_i = \frac{1}{2}(1-\varepsilon r_i)$ $i = 1,2,$. Then

$$\phi_1^{*\prime\prime} + \lambda f_2(\cdot,\phi_1^*) = \lambda f_2(\cdot,\phi_1^*) - \lambda f_1(\cdot,\phi_1^*) =$$

$$= \tfrac{1}{2}\varepsilon\lambda\phi_1^*(1-\phi_1^*)(r_2-r_1) \le 0 \; . \; (\not\equiv 0) \; .$$

Hence ϕ_1^* is a supersolution of the problem

$$\begin{cases} v'' + \lambda\, f_2(\cdot,v) = 0 \\ v(0) = 0 \quad v'(1) = 0 \,. \end{cases}$$

Since $v(x) \equiv 0$ is a subsolution it follows that there exists a solution w such that $0 \le w \le \phi_1^*$, and because neither 0 nor ϕ_1^* are solutions, $0 < w(x) < \phi_1^*(x)$ for $0 < x \le 1$. Because Problem (P") has only one solution by Proposition 2, $w = \phi_2^*$ and hence $\phi_2^*(x) < \phi_1^*(x)$ for $0 < x \le 1$.

COROLLARY. *Let* r *satisfy* H1 *and* H2 *and let* $\lambda \in \mathbb{R}^+$ *be fixed. Denote the solution of Problem* (P") *by* $\phi^*(\varepsilon)$. *Then* $\varepsilon_1 \ge \varepsilon_2$ *implies* $\phi^*(\varepsilon_1) \ge \phi^*(\varepsilon_2)$.

We now return to Problem (P). We can deduce from Proposition 2 that

$$C_s := \left\{(\lambda,\phi(\lambda))\,|\,\lambda>0\right\}$$

is an analytic curve of solutions with the symmetry property

$$(3.1) \qquad \phi(\lambda)(-x) = 1 - \phi(\lambda)(x)$$

such that

$$\phi(\lambda) \to \tfrac{1}{2} \quad \text{as} \quad \lambda \downarrow 0 \qquad \text{in} \quad C^2([-1,1]) \,,$$

and

$$\phi(\lambda)(x) \to \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x \in (-1,0) \end{cases} \qquad \text{as } \lambda \to \infty .$$

This means, in view of Theorem 1, that C_s is the subcontinuum of C consisting of the solutions with the symmetry property (3.1). Observe that all increasing solutions satisfying (3.1) belong to C_s .

In the following theorem we investigate the sign of the eigenvalue $\nu_k(\lambda)$ of the linearized problem

$$\text{(EV)} \qquad \begin{cases} -h'' - \lambda f_u(.,\phi(\lambda)\ h = \gamma(\lambda)h & -1 < x < 1 \\ h'(-1) = h'(1) = 0 \end{cases}$$

where $(\lambda,\phi(\lambda)) \in C_s$. If they are all positive for some $(\lambda,\phi(\lambda)) \in C_s$, we know that $\phi(\lambda)$ is asymptotically stable [3] and if one of them is negative that $\phi(\lambda)$ is unstable.

Let the eigenvalues ν_k be ordered so that $\nu_1 < \nu_2 < \nu_3 < \ldots$. Then due to the symmetry of f_u , $\nu_{2h} = \mu_h$ for $h = 1,2,\ldots$. We already know that $\mu_1(\lambda) > 0$ for all $\lambda > 0$. Hence $\nu_k \geq \nu_2 = \mu_1 > 0$ for all $k \geq 2$. Thus it remains to find the sign of ν_1 .

THEOREM 4. *Let* $(\lambda,\phi(\lambda)) \in C_s$. *Then there exists* λ_1 *and* λ_2 $(0<\lambda_1<\lambda_2<\infty)$ *such that*

(i) $0 < \lambda < \lambda_1$ *implies* $\nu_1(\lambda) < 0$

(ii) $\lambda_2 < \lambda < \infty$ *implies* $\nu_1(\lambda) > 0$,

where $\nu_1(\lambda)$ *denotes the principal eigenvalue of the Problem* (EV).

PROOF. (i) This part was proved in [6].

(ii) We begin with the observation that if $r^{(i)}$ $i = 1,2$ satisfy H1, H2 and $r^{(1)} > r^{(2)}$ on $(0,1]$, then the principal eigenvalues $\nu_1^{(1)}(\lambda)$ and $\nu_1^{(2)}(\lambda)$ satisfy the inequality:

$$\nu_1^{(1)}(\lambda) > \nu_1^{(2)}(\lambda) \quad \text{for all} \quad \lambda > 0 \,. \tag{3.2}$$

This follows from the fact that if $r^{(1)} > r^{(2)}$ then by Proposition 3, $\phi^{(1)}(x) > \phi^{(2)}(x)$ for all $x \in (0,1]$, and hence as follows from elementary computation $f_u^{(1)}(x,\phi^{(1)}(x)) < f_u^{(2)}(x,\phi^{(2)}(x))$ for all $x \in [-1,0) \cup (0,1]$. The variational characterization of the principal eigenvalue of Problem EV now gives the inequality (3.2). In view of the assumptions on r, there exists a constant $\delta > 0$ such that

$$xr(x) \geq \delta x^2 \quad \text{for} \quad -1 \leq x \leq 1 \,.$$

It follows from the above observation that it is sufficient to prove part (ii) for the function $r(x) = x$, $-1 \leq x \leq 1$, where the constant δ has been absorbed in the factor ε.

Let h_1 be the principal eigenfunction of Problem EV normalized so that $h_1 > 0$ on $[-1,1]$ and $\max\limits_{[-1,1]} h_1 = 1$. Then

$$\nu_1 h_1 = -h_1'' - \lambda f_u(\cdot,\phi(\lambda))h_1 \,. \tag{3.3}$$

If we multiply this equation by $\phi'(\lambda)$ and integrate over $(-1,1)$ we obtain, using the boundary conditions for h_1 and ϕ, and dividing by λ:

$$(3.4) \qquad \lambda^{-1} \nu_1(\lambda) \int_{-1}^{+1} \phi' h_1 dx = -f\Big(\cdot,\phi(\lambda)\Big) h_1 \Big|_{-1}^{+1}$$

$$+ \int_{-1}^{+1} f_x\Big(\cdot,\phi(\lambda)\Big) h_1 dx = -2f\Big(1,\phi(\lambda)(1)\Big) h_1(1) +$$

$$+ \varepsilon \int_0^{+1} \phi(\lambda)\Big(1-\phi(\lambda)\Big) h_1 dx .$$

Here we have also used the symmetry properties of f,ϕ and h. Integrating (3.3) over $(-1,1)$ and dividing by λ we obtain

$$\lambda^{-1} \nu_1(\lambda) \int_{-1}^{+1} h_1 dx = -2 \int_0^1 f_u\Big(\cdot,\phi(\lambda)\Big) h_1 \, dx =$$

$$= -2 \int_0^1 \phi(\lambda)(1-\phi(\lambda)) h_1 dx + 2 \int_0^1 (2\phi(\lambda)-1)(\phi(\lambda)-a) h_1 dx .$$

If we multiply this equality by $\frac{\varepsilon}{2}$ and add it to (3.4) we obtain

$$\lambda^{-1} \nu_1(\lambda) \int_{-1}^{+1} (\phi' + \tfrac{1}{2}\varepsilon) h_1 dx = -2f\Big(1,\phi(\lambda)(1)\Big) h_1(1) +$$

$$+ \varepsilon \int_0^1 \Big(2\phi(\lambda)-1\Big)\Big(\phi(\lambda)-a\Big) h_1 dx .$$

We assert that $h_1(x) \geq h_1(1)$ for $0 \leq x \leq 1$. Accepting this for the moment we find that

$$\lambda^{-1} \nu_1(\lambda) \int_{-1}^{+1} (\phi' + \tfrac{1}{2}\varepsilon) h_1 dx \geq h_1(1) \Big[-2f\Big(1,\phi(\lambda)(1)\Big) +$$

$$+ \varepsilon \int_0^1 \Big(2\phi(\lambda)-1\Big)\Big(\phi(\lambda)-a\Big) dx\Big] .$$

By Proposition 2, $\phi(\lambda)(x) \to 1$ as $\lambda \to \infty$ uniformly on any interval $[\xi,1]$ with $0 < \xi < 1$. Hence

$$\lim_{\lambda \to \infty} \left\{-2f\left(1,\phi(\lambda)(1)\right)+\varepsilon \int_0^1 \left(2\phi(\lambda)-1\right)\left(\phi(\lambda)-a\right)dx\right\} =$$

$$= \varepsilon \int_0^1 (1-a)dx = \frac{\varepsilon}{2}\left(1+\varepsilon \int_0^1 xdx\right) = \frac{\varepsilon}{2}(1+\frac{\varepsilon}{2}) > 0 \ .$$

Because the coefficient of $\nu_1(\lambda)$ is positive for all $\lambda > 0$, there exists a $\lambda_2 > 0$ such that $\nu_1(\lambda) > 0$ for all $\lambda > \lambda_2$.

It remains to prove that $h(x) \geq h(1)$ for $0 \leq x \leq 1$. One proof was given in [8]. Here we give another proof. Observe that $z = h_1'$ is a solution of the problem:

$$\begin{cases} -z'' - \lambda f_u z - \nu_1 z = \lambda f_{xu}\ h_1 \\ z(0) = z(1) = 0 \end{cases}$$

in which $f_{xu}(\cdot,\phi(\lambda)) = \frac{1}{2}\varepsilon[1-2\phi(\lambda)] < 0$. Note that by the definition of ν_1 the principal eigenvalue of the operator on the left hand side with Neumann boundary conditions is zero. Hence with Dirichlet boundary conditions it is positive, which implies that $z < 0$ on $(0,1)$, from which the result follows. This completes the proof of the Theorem 4.

Since $\nu_1(\lambda) < 0$ for all $\lambda > 0$ if $\varepsilon = 0$, one would expect that $\lambda_1,\lambda_2 \to \infty$ as $\varepsilon \to 0$. We shall show that this is indeed true. Define

$$\lambda_1^* = \sup\left\{\lambda>0 \middle| \nu_1 <0 \text{ on } [0,\lambda)\right\} .$$

Clearly λ_1^* will depend on ε. Thus, we shall write $\lambda_1^*(\varepsilon)$.

THEOREM 5. $\lambda_1^*(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$.

PROOF. Suppose to the contrary that there exist a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$ and a constant M such that $\lambda_1^*(\varepsilon_n) < M$, for all $n \geq 1$. Then there is a subsequence, still denoted by $\{\varepsilon_n\}$, and $\bar{\lambda} > 0$ such that

$$\lambda_1^*(\varepsilon_n) \to \bar{\lambda} \quad \text{as} \quad \varepsilon_n \to 0 .$$

Since ν_1 depends continuously on $(\lambda,\varepsilon), \nu_1(\lambda_1^*(\varepsilon_n)) = 0$ for all $n \geq 1$, and therefore, by (3.4)

$$f\Big(1,\phi(\lambda_n)(1)\Big)\, h_1\Big(\lambda_n(1)\Big) = \tfrac{1}{2}\varepsilon_n \int_0^1 \phi(\lambda_n)(1-\phi(\lambda_n))\, h_1(\lambda_n)dx ,$$

where we have written $\lambda_n = \lambda^*(\varepsilon_n)$. Hence

$$\left|f\Big(1,\phi(\lambda_n)(1)\Big)h_1(\lambda_n)(1)\right| \leq \frac{1}{2}\varepsilon_n \int_0^1 \frac{1}{4}h_1(\lambda_n)dx \leq \frac{1}{8}\varepsilon_n .$$

By using the continuity of $\phi(\lambda)(1)$ and $h_1(\lambda)(1)$ we get by taking the limit as $n \to \infty$:

$$f\Big(1,\phi(\bar{\lambda})(1)\Big)\, h_1(\bar{\lambda})(1) = 0 .$$

But $\phi(\bar{\lambda}) \in (0,\tfrac{1}{2})$ and because $a(1) \in (0,\tfrac{1}{2})$, $f(1,\phi(\bar{\lambda})(1)) \neq 0$. Since $h_1'(\bar{\lambda})(1) = 0$, we have $h_1(\bar{\lambda})(1) \neq 0$. Therefore we have a contradiction. This completes the proof of Theorem 5.

REMARKS. 1) If we define $\lambda_2^* = \inf\{\lambda>0 | \nu_1(\lambda)>0\}$, then $0 < \lambda_1^* < \lambda_2^* < \infty$ and $\nu_1(\lambda_1^*) = \nu_1(\lambda_2^*) = 0$. Since ν_1 is analytic in λ there are at most finitely many zeros of ν_1 in the interval $[\lambda_1^*, \lambda_2^*]$, and there is at least one zero where ν_1 changes sign. We denote by $\bar\lambda$ the first zero of ν_1 where ν_1 changes sign. It follows as in [7], that $(\bar\lambda, \phi(\bar\lambda))$ is a bifurcation point from which "emanates" a continuum of "nonsymmetric" solutions.

2) We are unable to prove that $\frac{d\nu}{d\lambda}(\bar\lambda) < 0$ which would imply the transervality condition needed for the theorem of bifurcation from simple eigenvalue. Since ν is analytic we only know that there exists an $m \in \mathbb{N}$ such that $\frac{d\nu^n}{d\lambda^n}(\lambda) = 0$ for $n \leq 2m$ and $\frac{d\nu^{2m+1}}{d\lambda^{2m+1}}(\lambda) > 0$.

3)Concerning the nonsymmetric case, we just mention the following result which suggest that the branch of stable solutions becomes disconnected from C when $h \neq 0$.

PROPOSITION 6. *There is no* $\delta > 0$ *such that there exists a* C^1 *function* $w: [-\delta,\delta) \times (\bar\lambda-\delta, \bar\lambda+\delta) \to C^2([-1,1])$ *which satisfies*

i) $w(0,\lambda) = C(\lambda)$ $\lambda \in (\bar\lambda-\delta, \bar\lambda+\delta)$

ii) $w(h,\lambda)$ *is a solution of Problem* (P) *for each* $h \in (-\delta,\delta)$.

PROOF. If such a δ would exist, then $z = \frac{\partial w}{\partial h}(0,\bar\lambda)$ would satisfy the equation:

$$\text{(E)} \qquad \begin{cases} -z'' - \bar\lambda f_u(\cdot,\phi(\bar\lambda),h=0)z = \bar\lambda\phi(\bar\lambda)(1-\phi(\bar\lambda)) \\ z'(-1) = z'(1) = 0 \ . \end{cases}$$

This is obtained by differentiating (P) with respect to h , at $h = 0$. But at $(\bar\lambda, \phi(\bar\lambda))$, the principal eigenvalue $\nu_1(\bar\lambda)$ vanishes and the corresponding eigenfunction h_1 does not change sign. Thus $\int_{-1}^{+1} \bar{\lambda}\phi(1-\phi)h_1 dx \neq 0$ which contradicts the necessary orthogonality condition $\int_{-1}^{+1} \bar{\lambda}\phi(1-\phi)h_1 dx = 0$ in order to have existence of a solution z. This completes the proof of the proposition.

Note that

$$\frac{\partial u}{\partial h}\left(\lambda, h=0\right)(x) > 0 \qquad x \in [-1,1] , \tag{3.5}$$

$\bar{\lambda} < \lambda < \bar{\bar{\lambda}}$ (next zero of $\nu_1(.)$, possibly $+\infty$) and there is $\bar{\delta} > 0$ such that

$$\frac{\partial u}{\partial h}\left(\lambda, h=0\right)(x) < 0 \qquad x \in [-1,1] , \ \bar{\lambda} - \bar{\delta} < \lambda < \bar{\lambda} . \tag{3.6}$$

The existence of $\frac{\partial u}{\partial h}(\lambda,.) \in C^2[-1,1]$ follows from the implicit function theorem and $z = \frac{\partial u}{\partial h}$ satisfies (E) for $\lambda \in (\bar{\lambda}-\delta_1,\bar{\lambda}) \cup (\bar{\lambda},\bar{\bar{\lambda}})$ for some $\delta_1 > 0$. Since the right-hand side of (E) is positive, and the operator on the left-hand side is coercive $(\nu_1 > 0)$ for $\lambda \in (\bar{\lambda},\bar{\bar{\lambda}})$, (3.5) follows. The inequality (3.6) is a consequence of the antimaximum principle [2].

4. AN EXAMPLE IN WHICH $f(.,u)$ IS PIECEWISE LINEAR

In this section we shall assume that $f(x,u)$ has the following form

$$f(x,u) := \begin{cases} -u & 0 \leq u < a(x) \\ 0 & u = a(x) \\ 1-u & a(x) < u \leq 1 , \end{cases}$$

where

$$a(x) := \tfrac{1}{2}(1-\varepsilon x+h) \qquad 0 < \varepsilon < 1 , \ |h| < \min\{\varepsilon, 1-\varepsilon\}. \tag{4.1}$$

Thus f is not continuous in u but it still has the properties

$$f(x,0) = f(x,1) = 0$$

$$f(x,u) < 0 \quad \text{for} \quad x \in (-1,1)\ ,\ 0 < u < a(x)$$

$$f(x,u) > 0 \quad \text{for} \quad x \in (-1,1)\ ,\ a(x) < u < 1\ .$$

We consider the problem:

Find $(\lambda,u) \in \mathbb{R}^+ \times W^{2,\infty}(-1,1)$ such that

$$(\tilde{P}) \qquad \begin{cases} -u'' = \lambda f(.,u)\ ,\ u' > 0 \quad \text{on} \quad (-1,1) \\ u'(-1) = u'(+1) = 0\ . \end{cases}$$

Thus, u satisfies

$$(4.2a) \qquad -u'' = -\lambda u \qquad \text{if} \quad u(x) < a(x)$$

$$(4.2b) \qquad -u'' = \lambda(1-u) \quad \text{if} \quad u(x) > a(x)\ .$$

Because u is increasing and a is decreasing there exist a unique $\xi \in (-1,1)$ such that $u(x) < a(x)$ on $[-1,\xi)$ and $u(x) > a(x)$ on $(\xi,1]$. Thus by (4.2a,b) and the boundary conditions

$$u(x) = \alpha \cosh \sqrt{\lambda}(x+1) \qquad \text{if} \quad -1 \le x \le \xi$$

$$u(x) = 1 - \beta \cosh \sqrt{\lambda}(x-1) \quad \text{if} \quad \xi \le x \le 1\ ,$$

where $\alpha,\beta \in \mathbb{R}$.

Since $u \in C^1([-1,1])$ we must impose at $x = \xi$ the conditions

(4.3a) $\quad \alpha \cosh \sqrt{\lambda}(\xi+1) = 1 - \beta \cosh\sqrt{\lambda}(\xi-1) = a(\xi)$

(4.3b) $\quad \alpha \sqrt{\lambda} \sinh \sqrt{\lambda}(\xi+1) = -\beta \sqrt{\lambda} \sinh \sqrt{\lambda}(\xi-1)$.

Using the definition of $a(\xi)$, and eliminating α and β , we obtain

(4.4) $\quad (1+h-\varepsilon\xi) \tanh \sqrt{\lambda}(\xi+1) + (1-h+\varepsilon\xi) \tanh \sqrt{\lambda}(\xi-1) = 0$.

This leads to

(4.5) $\quad \sinh 2\sqrt{\lambda}\, \xi = (\varepsilon\xi-h)\sinh 2\sqrt{\lambda}$

or, when we introduce the function,

$$\Phi(t) := \begin{cases} \dfrac{\sinh t}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 , \end{cases}$$

to

(4.6) $\quad \xi\Phi(2\sqrt{\lambda}\xi) = (\varepsilon\xi-h)\Phi(2\sqrt{\lambda})$.

I. THE SYMMETRIC CASE (h=0)

Equation (4.6) now reduces to

(4.7) $\quad \xi\left\{\Phi(2\sqrt{\lambda}\xi)-\varepsilon\Phi(2\sqrt{\lambda})\right\} = 0$.

Plainly, $\xi = 0$ is a solution for all $\lambda > 0$. The corresponding values of α and β can be computed from (4.3a). They are

$$\alpha = \beta = 1/(2 \cosh \sqrt{\lambda})$$

Thus, there exists a continuous branch of symmetric solutions.

Other solutions of (4.7) must satisfy

$$(4.8) \qquad P(\lambda,\xi) := \frac{\Phi(2\sqrt{\lambda}\,\xi)}{\Phi(2\sqrt{\lambda})} = \varepsilon$$

where we have divided (4.7) by $\xi\Phi(2\sqrt{\lambda})$. Because P has the properties

(i) $P(\lambda,t) = P(\lambda,-t)$ for all $t \in \mathbb{R}$;

(ii) $P(\lambda,0) = 1/\Phi(2\sqrt{\lambda})$;

(iii) $t\,\frac{\partial P}{\partial t}(\lambda,t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$

it follows that (4.8) has exactly two solutions $\pm\,\bar{\xi}(\lambda,\varepsilon)$ if and only if $\varepsilon > 1/\Phi(2\sqrt{\lambda})$, i.e. when $\lambda > \lambda_0(\varepsilon)$ where $\Phi(2\sqrt{\lambda_0(\varepsilon)}) = 1/\varepsilon$.

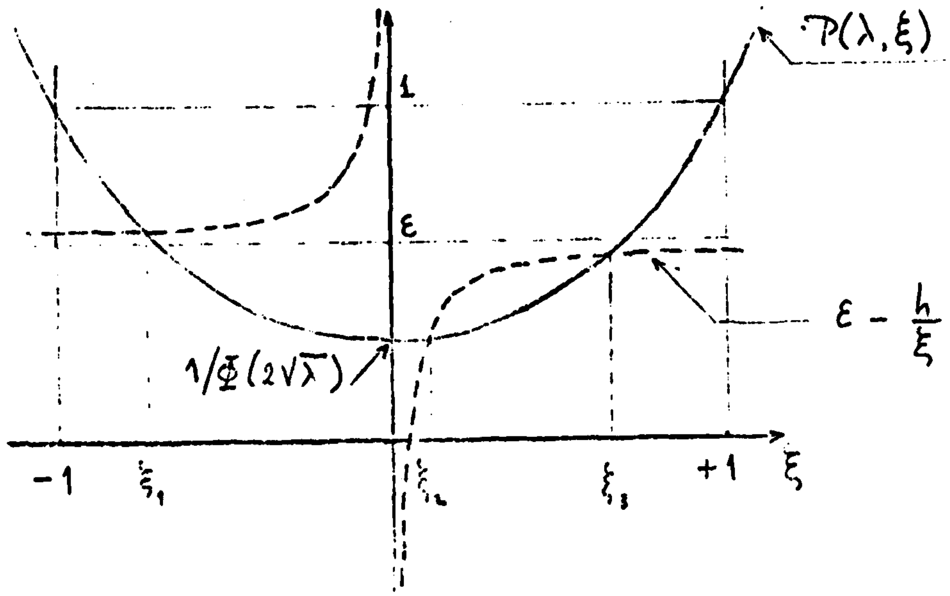

Fig. 1 The case $\lambda > \lambda_0$

The corresponding values of α and β can again be determined from (4.3a).

Thus, summarizing we have found

1. For each $\lambda > 0$, there exists exactly one symmetric solution, characterized by $\xi = 0$.

2. For $\lambda > \lambda_0(\varepsilon)$ there exist exactly three solutions: the symmetric one ($\xi=0$) and two non-symmetric ones characterized by $\xi = \pm\,\xi(\lambda)$. It is readily seen from Fig. 1 that

$$\lambda_0(\varepsilon) \to \begin{cases} 0 & \text{as} \quad \varepsilon \to 1 \\ \infty & \text{as} \quad \varepsilon \to 0 . \end{cases}$$

II. THE ASYMMETRIC CASE ($h \neq 0$)

It follows from (4.5) that $\xi = 0$ implies $\lambda = 0$. Thus when $\lambda > 0$, we may divide by ξ, and solutions must satisfy the equation

$$(4.9) \qquad P(\lambda,\xi) = \varepsilon - \frac{h}{\xi} .$$

We consider the case $h > 0$.

One verifies that for each $\lambda > 0$, the function $\xi \to P(\xi,\lambda)$ is strictly convex and even. Thus $P(0,\lambda) = 1/\Phi(2\sqrt{\lambda})$ is the minimum of $P(\xi,\lambda)$.

The function $\xi \to \varepsilon - (h/\xi)$ is strictly increasing for $\xi \in [-1,0) \cup (0,1]$. Since by (4.1)

$$\varepsilon + h < P(-1,\lambda) = 1$$

and

$$\lim_{\xi \uparrow 0} \varepsilon - \frac{h}{\xi} > P(0,\lambda) = 1/\Phi(2\sqrt{\lambda})$$

there exists for each $\lambda > 0$ and $h \in (0,1-\varepsilon)$ exactly one solution $\xi_1(\lambda,h) \in [-1,0)$ of equation (4.9). Since

$$\frac{\partial}{\partial\xi}\left[P(\xi,\lambda)-\varepsilon+\frac{h}{\xi}\right] = \frac{\partial P}{\partial\xi}(\xi,\lambda) - \frac{h}{\xi^2} < 0 \quad \text{for} \quad \xi < 0$$

the implicit function theorem implies that ξ_1 as a function of λ is smooth and strictly decreasing.

For $\xi \in (0,1]$, both the left-hand side and the right-hand side of (4.9) are increasing in ξ, but $\xi \to P(\xi,\lambda)$ is strictly convex and $\xi \to \varepsilon - (h/\xi)$ is strictly concave. Moreover, we have

$$\lim_{\xi\downarrow 0} P(\xi,\lambda) = 1/\Phi(2\sqrt{\lambda}) > \lim_{\xi\downarrow 0} (\varepsilon-\frac{h}{\xi})$$

and

$$P(1,\lambda) = 1 > \varepsilon - h \qquad \text{by (4.1).}$$

Thus equation (4.9) has zero, one or two solutions.

Observe that P as a function of λ is strictly increasing, because

$$\frac{\partial}{\partial\lambda} P(\xi,\lambda) = \frac{1}{2\lambda} P(\xi,\lambda) \left\{F(2\sqrt{\lambda}\xi)-F(2\sqrt{\lambda})\right\}$$

where $F(t) = t\Phi'(t)/\Phi(t)$ and

$$F'(t) = \Phi^{-2}\left(\Phi\Phi'+t\Phi\Phi''+t\Phi'^2\right) =$$

$$= \tfrac{1}{4}(t\Phi)^{-2}\left(\sinh 2t-2t\right) > 0 \quad \text{for} \quad t > 0 \ .$$

For $\lambda < \lambda_0(\varepsilon)$ there is no solution since

$$\inf_{\xi\in(0,1]} P(\xi,\lambda) > \varepsilon > \varepsilon - h = \sup_{\xi\in(0,1]} (\varepsilon-\frac{h}{\xi}) .$$

Because $P(\xi,\lambda) \to 0$ as $\lambda \to \infty$ and $\xi \in (0,1)$ fixed, there exist two solutions of (4.9) for λ large. Remembering that $\partial P(\xi,.)/\partial\lambda < 0$ we conclude that there exists a $\lambda^* \in (\lambda_0,\infty)$ such that there exists no solution of (4.8) if $\lambda < \lambda^*$, exactly one of $\lambda = \lambda^*$ and exactly two of $\lambda > \lambda^*$. Since

$$\frac{\partial}{\partial\lambda}\left[P(\xi,\lambda)-\varepsilon+\frac{h}{\xi}\right] = \frac{\partial}{\partial\lambda} P(\xi,\lambda) < 0$$

these solutions can be smoothly parametrized by $\xi \in (0,1)$.

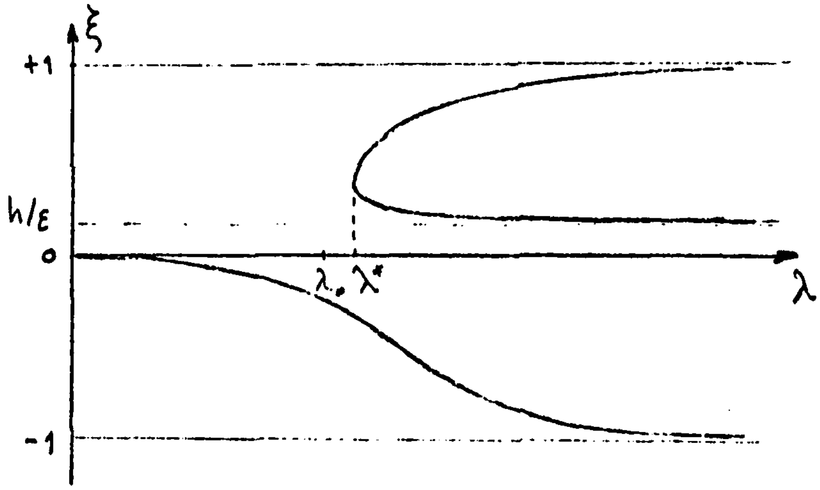

Fig. 2 The bifurcation diagram (h>0)

For $h < 0$ the situation is entirely analogous. This case can be reduced to the former by substituting $h \to -h$ and $\xi \to -\xi$ in (4.9).

APPENDIX A

In this Appendix we state and prove a version of a global function theorem which was used in section 2. This variant is a slight extension of Theorem 6.2 of [9]. For the sake of completeness we give a proof here.

THEOREM A. *Let* E *be a real Banach space and let* $F: \mathbb{R} \times E \to E$ *satisfy*

a) $F(0,0) = 0$

b) F *is continuous*

c) $G: \mathbb{R} \times E \to E$ *defined by* $G(\lambda,w) = w - F(\lambda,w)$ *maps bounded subsets of* $\mathbb{R} \times E$ *into relatively compact subsets of* E.

d) F *possess a partial Fréchet derivative with respect to* w *at* (0,0) : $dF_w(0,0)$ *and* $dF_w(0,0) \in \mathrm{Isom}(E,E)$.

e) *If* J *denotes the set of solutions of*

$$F(\lambda,w) = 0 \quad \text{in} \quad \mathbb{R} \times E \tag{1}$$

and if D *denotes the maximal connected subset of* J *which contains* (0,0), *then*

$$D \cap (\{0\}\times E) = (0,0) .$$

Then D *is unbounded in* $\mathbb{R}^+ \times E$ *and in* $\mathbb{R}^- \times E$.

f) *If moreover* $F \in C^1(\mathbb{R}\times E;E)$, *then there exists* $\delta > 0$, $\lambda_0 > 0$ *and a* C^1 *function* $\lambda \to z(\lambda) \in E$ *such that* $(\lambda,u) \in J$ and $|\lambda| > \lambda_0$, $\|u\| < \delta$, *imply* $u = z(\lambda)$. *The function* z *is* C^m *(resp. analytic) whenever* F *is* C^m *(resp. analytic).*

PROOF. If the conclusion of the theorem does not hold, then either $C^+ := C \cap (\mathbb{R}^+ \times E)$ or $C^- := C \cap (\mathbb{R}^- \times E)$ is bounded. Assume C^+ bounded. It follows from b) and c) that C^+ is a compact metric space under the induced topology from $\mathbb{R} \times E$. It follows from d) that $(0,0)$ is an isolated solution of (1) in $\{0\} \times E$. Since C^+ is compact it follows from e) that there exists a δ-neighbourhood of C^+ in $\mathbb{R} \times E$ denoted by U such $U \cap (\{0\} \cap E) \cap J = (0,0)$. Let $K := \bar{U} \cap J$ where $\bar{U}$ denotes the closure of U in $\mathbb{R} \times E$. Since U is bounded, $\bar{U}$ is bounded and thus $\bar{U} \cap J$ is totally bounded. $\bar{U}$ and J are closed, thus K is compact metric. By construction $\partial U \cap C^+ = \emptyset$. Hence by [9], there exist disjoint compact subsets $A, B \subset K$ such that $C^+ \subset A$, $\partial U \cap J \subset B$ and $K = A \cup B$. Let $\mathcal{O}$ be a ρ-neighbourhood of A where ρ is less than the distance from A to B and less than the distance from A to ∂U. Note that these distances are positive since A is compact and B, ∂U are closed, $A \cap B = \emptyset$ and $A \cap \partial U = \emptyset$. Hence $C^+ \subset \mathcal{O}$, $\partial\mathcal{O} \cap J = \emptyset$ $\mathcal{O}$ is bounded and $\mathcal{O} \cap (\{0\} \times E) = (0,0)$, $\mathcal{O} \cap (\{\lambda\} \times E) = \emptyset$ for λ large enough. Define $O_\lambda := \{u \in E \mid (\lambda, u) \in \mathcal{O}\}$. From b) and c), it follows that the Leray-Schauder degree of F is well-defined on O_λ: $d(F, O_\lambda)$. For λ large enough, $d(F, O_\lambda) = 0$ since $O_\lambda = \emptyset$. For $\lambda = 0$ $d(F, O_0) = \text{index}(F(0,.),0) = \text{index}(dF_w(0,0),0) \neq 0$ since $dF_w(0,0) \in \text{Isom}(E,E)$. But this contradicts the homotopy-invariance of the Leray-Schauder degree (see Lemma 1.8 of [9]). Thus C^+ is unbounded in $\mathbb{R}^+ \times E$. Similarly C^- is unbounded in $\mathbb{R}^- \times E$.

The last assertion follows directly from the local implicit function theorem.

APPENDIX B

The aim of this Appendix is to prove that stable stationary solutions of (D) are increasing when a is nonincreasing. This generalizes an earlier result of Chafee [1]. It is known that if (λ,u) is a stable stationary solution of (D), then the principal eigenvalue ν_1 of (EvN)

$$\text{(EvN)} \begin{cases} -h'' - f_u(\cdot,u)h = \nu h \\ h'(-1) = h'(1) = 0 \end{cases}$$

is nonnegative. Thus the principal eigenvalue of (EvD)

$$\text{(EvD)} \begin{cases} -h'' - f_u(\cdot,u)h = \nu_D h \\ h(-1) = h(1) = 0 \end{cases}$$

is strictly positive: $\nu_{D,1} > \nu_1 \geq 0$.

PROPOSITION. *Let* a *satisfy* A1,A2 *and let*

A3' $\qquad a'(x) \leq 0 \quad x \in [-1,1]$.

Let (λ,u) *be a stable solution of* (P) *satisfying* $0 \leq u(x) \leq 1$, $x \in [-1,1]$, $\lambda > 0$. *Then either* u *is identically constant*, $u \equiv 0$, $u \equiv 1$ *or, when* a *is not a constant*, u *is strictly increasing*.

PROOF. $v = u'$ satisfies

$$\begin{cases} -v'' - \lambda f_u(\cdot,u)v = \lambda f_x = \lambda u(1-u)(-a') \geq 0 \\ v(-1) = v(1) = 0 . \end{cases}$$

Since $\nu_{D,1} > 0$, the operator on the left-hand side is coercive. If u is not a constant, then it follows from the maximum principle that $0 < u(x) < 1$, $x \in [-1,1]$ and thus the right-hand side is nonnegative $(\neq 0)$. Then $u'(x) = v(x) > 0$ on $(-1,1)$. If u is a constant it can only be a trivial solution if a is not a constant, and, if a is a constant, and $u = a$, it is easily seen to be unstable.

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We discuss the nonlinear eigenvalue problem

$$(P)\quad \begin{cases} u'' + \lambda f(x,u) = 0 & -1 < x < 1,\ \lambda \geq 0 \\ u'(-1) = u'(+1) = 0, & 0 \leq u(x) \leq 1 \end{cases}$$

where

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20. ABSTRACT - cont'd.

$$f(x,u) = u(1 - u)[u - a(x)]$$

and

$$a(x) = \frac{1}{2}[1 - \varepsilon r(x) + h] \qquad \varepsilon \geqq 0, \quad h \in \mathbb{R}$$

with $r(-x) = -r(x)$ and $r' \geqq 0$.

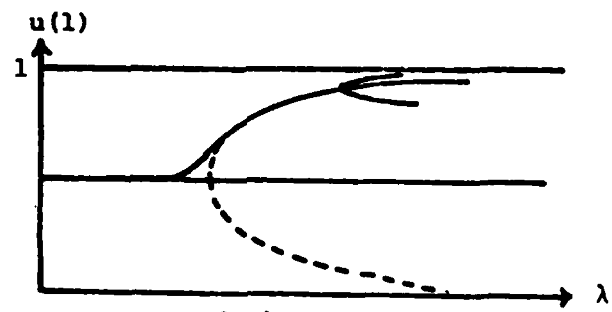
For $\varepsilon = h = 0$ the solution of Problem P is well known, and every solution, except $u = 0$ and $u = 1$ is unstable with respect to the corresponding parabolic problem.

We show how the branch of increasing solutions changes as ε becomes positive, and acquires a bifurcation point $(\bar{\lambda},\bar{u})$. beyond which this branch becomes stable. If h becomes nonzero as well, this bifurcation point is shown to break up.

As an illustration we consider an example in which the branch of increasing solutions can be computed. Here

$$f(x,u) = \begin{cases} -u & 0 \leqq x < a(x) \\ 1 - u & a(x) < u \leqq 1 \end{cases}$$

where $a(x)$ is given above.



$\varepsilon > 0, \ h = 0$

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